Gravitational Lensing Shear Simulations Using Fast Fourier Transform

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Abstract

Gravitational lensing is a natural phenomenon often used in the study of dark matter. Numerical studies of gravitational lensing often make use of the Fast Fourier Transform to relate mass density and image distortions. However, such convolution calculations have undesired effects on their results that must be understood and corrected. Two mass density models are presented here to demonstrate such inaccuracies.

Introduction

According to the currently accepted model of the structure and evolution of the universe, known as the Concordance Model, dark matter accounts for roughly 25% of the total energy density of the universe and about 85% of all matter. Though constituting such a large portion of the universe, dark matter has eluded more direct probes its distribution until recent years, as dark matter interacts too weakly to be seen in any wavelength of the electromagnetic spectrum with current instruments. However, gravitational lensing solves the problem of studying matter that cannot be seen.

Gravitational lensing uses measurements of distortions of background images to understand the distribution of foreground mass between the observer and the background sources. Image distortion occurs either as isotropic image focusing, known as convergence, or anisotropic stretching, called shear. Lensing situations can be broken down into three groups: strong lensing, in which image distortions are noticeable as multiple images and rings, weak lensing, in which image distortions are only noticeable through statistical analysis of ellipticities over a large set of objects, and microlensing, in which strong lensing occurs but images cannot be resolved.

Both numerical simulations and observational studies of weak gravitational lensing often rely on the numerical technique of Fast Fourier Transform (FFT). However, FFT has some restrictions that must be understood when used to understand gravitational lensing results. Kaiser & Squires 1996 compare different methods of numerical techniques and demonstrate the noisiness in numerical results produced with FFT. In order to understand some of the effects FFT has on lensing simulations, two test cases are presented here, for which analytical results exist for comparison.

Overview of Lensing Properties

Though the gravitational effects of mass had been studied according to Newton's Law of Universal Gravitation since the late 18th century, the full extent of gravity's influence on space and time, the latter being a quite important breakthrough, was not fully understood until gravity was put into the context of space-time distortion under Einstein's General Theory of Relativity. One result of the new theory of gravitation is that gravity bends the path of light rays, not just matter. Therefore, a foreground mass will distort the image of a background object if the light rays pass sufficiently close to the foreground mass.

Lensing can be understood in a couple of ways. First, once could study the individual light rays to measure the angle of deviation from the original path that the foreground mass imparts on the passing rays. This angle is known as the deflection angle. This method is quite convenient for reproducing distorted images and is simple to setup and evaluate numerically[2]. However, another understanding of lensing proves a better match with observational techniques.

Image distortion can be directly understood through convergence and shear. Such a description has advantages over deflection angles. First, shear, or a reduced form of shear, is directly measured in astrophysical images, such as those taken by the Hubble Space Telescope. Therefore, even when used in a theoretical manner, one is rooted in observational technique, helping to bridge the possible experimental-theoretical gap. Another advantage to shear is its connection, though indirect, to convergence.

Convergence is directly proportional to the projected mass density of the foreground mass. Therefore, if one could measure the convergence, a 2-dimensional mass density profile would be obtained automatically. Convergence, represented by the Greek letter kappa, is given by the relationship,

$$\kappa(x_1, x_2) = \frac{\Sigma(x_1, x_2)}{\Sigma_{cr}} \qquad (1),$$

where Σ represents the mass density and Σ_{cr} is the so called critical density, a scalar function of the background and foreground redshifts. Variables x_1 and x_2 are the 2-dimensional coordinates; this naming convention will be used throughout the paper.

Shear is a measure of the change in ellipticity of an image. This quantity can be measured in lensing images of galaxies. In fact, weak lensing has only been recently studied due to the fact that one must be able to gain statistics on a large ensemble of galaxies, for which there is little distortion and the original ellipticity is not known. One natural question then follows: how do you measure shear if you do not know the original ellipticity? First, there is no reason to assume ellipticities of separated galaxies are correlated. Therefore, the average ellipticity should be zero. Secondly, the distortion of images tends to align the semi-major axis tangentially to a circle drawn around a peak in the mass density. Therefore, a collection of tangentially aligned galaxies suggest a concentration of mass. These shear measurements then can be related back to the convergence, and therefore the mass density.

All lensing quantities, including deflection angle, can be related to a scalar field defined throughout the lensing plane, known as the lensing potential. This is analogous to more familiar gravitational and electrostatic potentials. For a given potential, deflection angle is the gradient and convergence is half of the Laplacian. Shear, however, is a combination second derivatives of the potential given in two components,

$$\gamma_1 \equiv \frac{1}{2}\left(\frac{\partial^2\psi}{\partial {x_1}^2} - \frac{\partial^2\psi}{\partial {x_2}^2}\right) \quad (2)$$

$$\gamma_2 \equiv \frac{\partial^2\psi}{\partial x_1 \partial x_2} \quad (3),$$

where shear and potential are represented by Greek letters gamma and psi, respectively[3].

Shear Simulation

With numerical shear values readily available for arbitrary mass distributions, one could easily study various mass profiles, fitting such distributions with observational data. It is therefore useful in the study of dark matter within galaxies and galaxy clusters to have such numerical shear maps.

As was previously discussed, for a given foreground mass distribution, there exists a scalar lensing potential, describing the gravitational distortion effects on background images. Both mass density, via convergence, and shear are combinations of second derivatives of this lensing potential. The relationship between convergence and shear can be seen more clearly with the use of Fourier Transforms. This can be done by defining the potential in terms of a Fourier transform,

$$\psi(x_1, x_2) \equiv \int \tilde{\psi}(k_1, k_2) e^{i\vec{k}\cdot\vec{x}} d^2k \quad (4)$$

where tilde represents a function in Fourier space, with conjugate components, generally referred to as wave numbers, k_1 and k_2. With this definition of the potential, differentiation becomes multiplication in Fourier space, as the only factor containing "x" in the right hand side of Equation 4 is the exponential function. Convergence and shear can then be related to each other in a linear way, when calculations are preformed in Fourier space. With the computational technique of discrete Fourier transformation using the Fast Fourier Transform algorithm[4], the computationally costly calculation over the entire grid can now be performed in a relatively quick manner, O(N *log* N).

In the FFT method, mass particles are distributed on a grid according to the given mass distribution. In the calculations presented here, convergence values are calculated directly for each point on the grid, since distributions presented here, in order to verify computational veracity, are always analytical. Since the FFT method requires an N by N convergence grid, there is no need to leave some points empty or estimated by binning particles when the distribution is known.

After the convergence grid is created, the grid is transformed via the FFT. These transformed values are then multiplied by the appropriate wave numbers, a function which is generally referred to as the kernel. It is worth noting that some FFT algorithms, such as the one described in *Numerical Recipes*[4], give Fourier conjugate values arranged in order from the zero frequency to the highest positive frequency then from second most negative to just below zero. Therefore, one must be careful when using such an algorithm that the correct wave numbers are used for each transformed value. One solution to this possible source of error is to simultaneously transform the kernel function from its real-space corresponding function. Doing this extra transformation insures a correct correspondence of wave numbers, via an inner product of the two transforms, while at the same time only costing additional computing time of O(N *log* N). Finally, the inverse Fourier transform of this product gives the shear values in the lensing plane, once correctly normalized.

Two mass distributions, which have analytical shear values, are presented with FFT calculated shear values, a uniform elliptical distribution and a Singular Isothermal Ellipse (SIE)[5]. Figures 1 and 2 show the results of the uniform ellipse and SIE, respectively. Both sets of calculations are preformed at background and foreground redshifts of 0.7 and 0.3, respectively. The total mass of the uniform ellipse is $3 * 10^{14}$ Solar Masses, with pixel lengths representing 0.39 arcseconds. The Singular Isothermal Ellipse is distributed according to the relation,

$$\kappa(x,\theta) = \frac{\alpha}{2x}\left[1 + \frac{3\epsilon}{2}\cos(2\theta)\right] \quad (5).$$

Here the values of alpha and epsilon are 13 arcseconds and 0.3, respectively, in agreement with literature[5]. Also grid sizes were chosen to be 52 arcseconds wide.

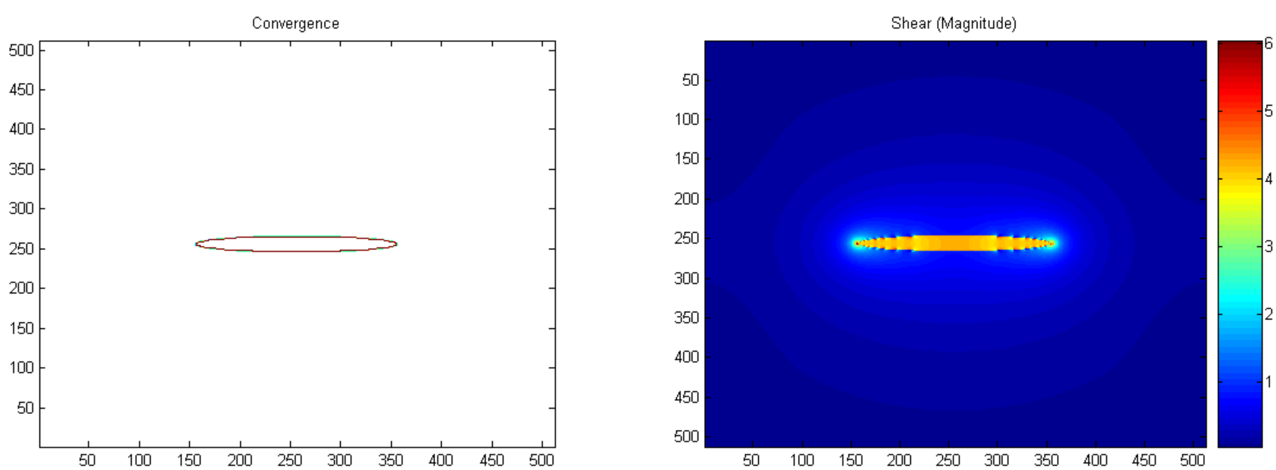

Figure 1. The convergence and simulated shear values for the uniform ellipse. These calculations are preformed at background and foreground redshifts of 0.7 and 0.3, respectively. Grid positions are listed in pixels, with 1 pixel equaling 0.39 arcseconds.

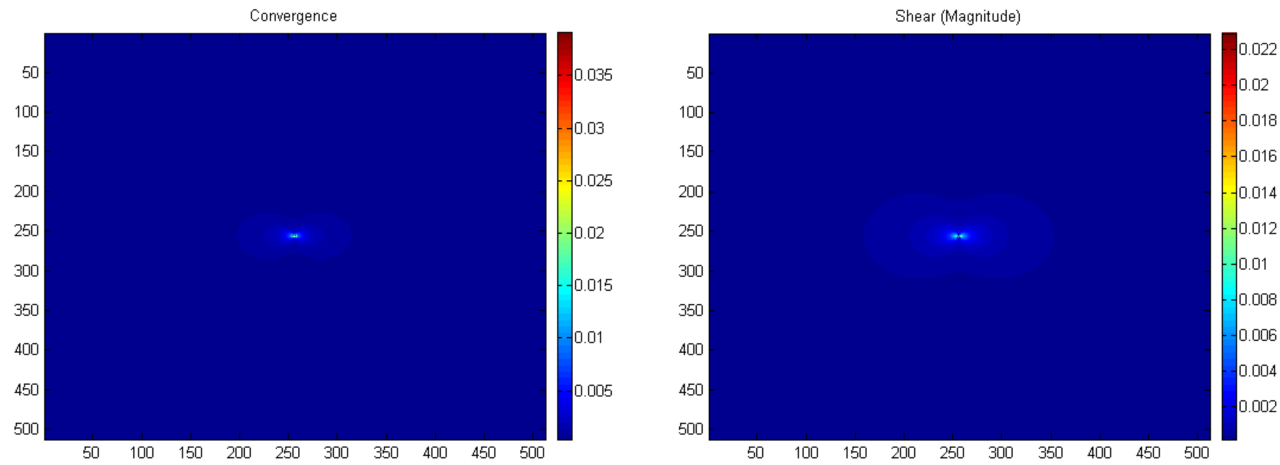

Figure 2. The convergence and simulated shear values for the Singular Isothermal Sphere. These calculations are preformed at background and foreground redshifts of 0.7 and 0.3, respectively. Grid positions are listed in pixels, with the grid width equaling 52 arcseconds.

Result Verification

Anytime numerical simulations are used to represent physical phenomena, one must ensure results agree with actual physical values. Both distributions presented here were chosen for that purpose, since they have clear analytical shear functions.

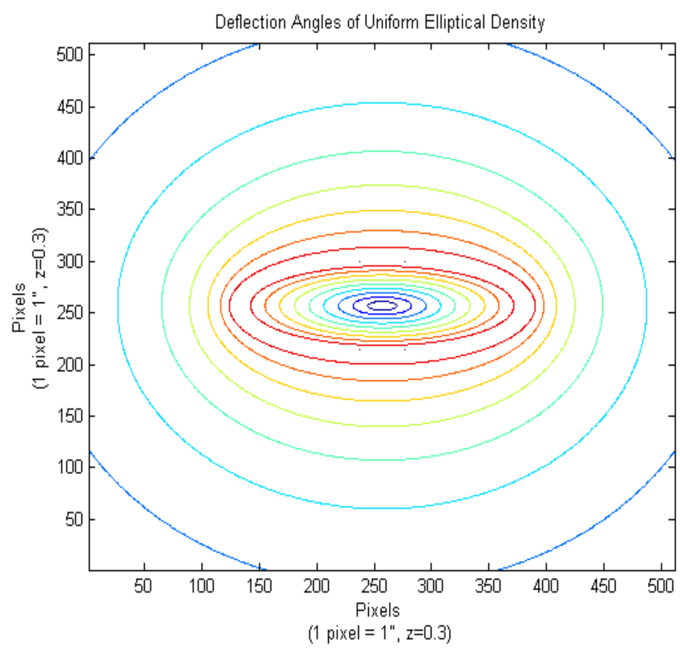

Figure 3. Lines of equal deflection angle values for a uniform ellipse with a ratio of semi-major to semi-minor axes of 10. This particular calculation was performed at a redshift of 0.3 as well.

Uniform elliptical mass distributions have deflection angles that form ellipses of equal value, confocal with the mass distribution[6]. An example of such a lens is given in Figure 3. With the deflection angles known, one can calculate the inverse magnification, which is also directly related to the shear and convergence[3]. This comparison can be used to show correlation between the analytical and numerical results. Another advantage to the inverse magnification comparison is the fact that outside of the mass distribution, only the shear contributes, as the convergence vanishes. Both analytical and numerical values of the inverse magnification are shown in Figure 4.

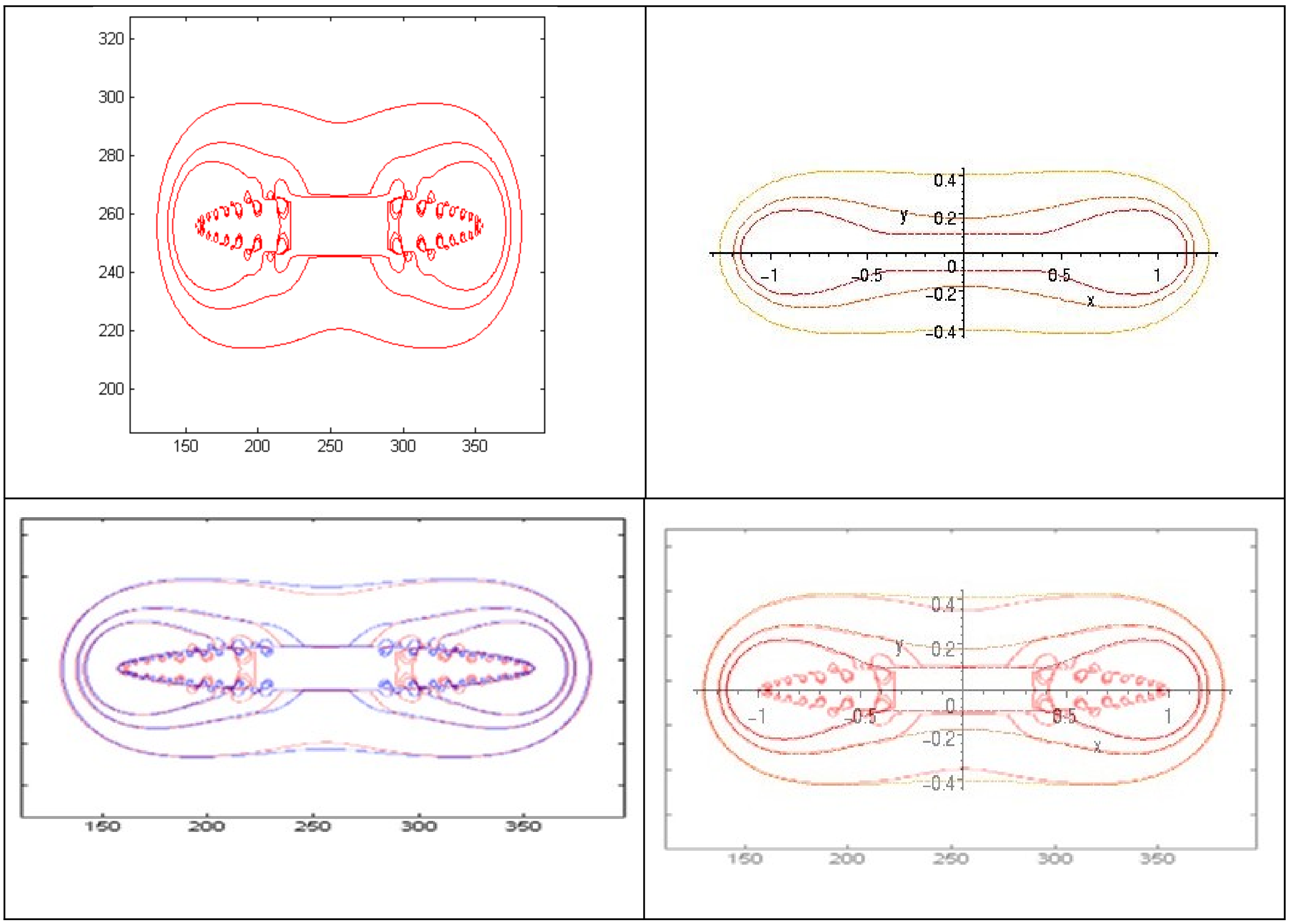

Figure 4. Numerical inverse magnification calculations for two different resolutions with analytical values. The upper images are the numerical (left) and analytical (right) values. The lower images show overlays of double resolution (left), in blue, and combined analytical and numerical values (right). Distances on the axes in the analytical plot are in terms of the semi-major axis length. Axes lengths in numerical results are in pixels values, where 1 pixel = 0.39 arcseconds.

Each inverse magnification plot has three contour lines for inverse magnification values of 0, 0.3 and 0.6, where the zero value is shown by the inner most contour and 0.6 is the outer most contour. The zero-value contour corresponds to the so called critical curves, where magnification diverges. The top two images in Figure 4 show the numerical (left) and analytical (right) results. The axes of the numerical results are measured in pixels, where one pixel equals 0.39 arcseconds, whereas the axes of the analytical results are in units of the semi-major axis length. For better comparison, the bottom images show overlays of different computation parameters. The left image shows the influence of resolution on the FFT computation, where the blue curves were produced with double the resolution of the red curves. The right image overlays the top two images of Figure 4.

From these results, one can see a few setbacks of the FFT method of shear calculation. The numerical/analytical overlay image shows the FFT method has problems at both the grid boundary and a mass density boundary. All numerical results show the outline of the mass ellipse traced out by the critical curve. However, this result is less problematic, as such a sharp edge does not exist in galaxy clusters to be studied.

The grid boundary problem, on the other hand, is critical. The 0.3 and 0.6 lines of the numerical/analytical overlay image show the FFT calculation results in an underestimate of the shear values in the case of the uniform ellipse. As will be more strongly reinforced with the SIE, this effect is believed to be due to a grid boundary problem associated with the FFT. The mass to shear relationship assumes a periodic mass function. The convergence clearly is not periodic. As a result, the edges are incorrectly affected by the opposite side of the grid. A potential solution to this problem is discussed later.

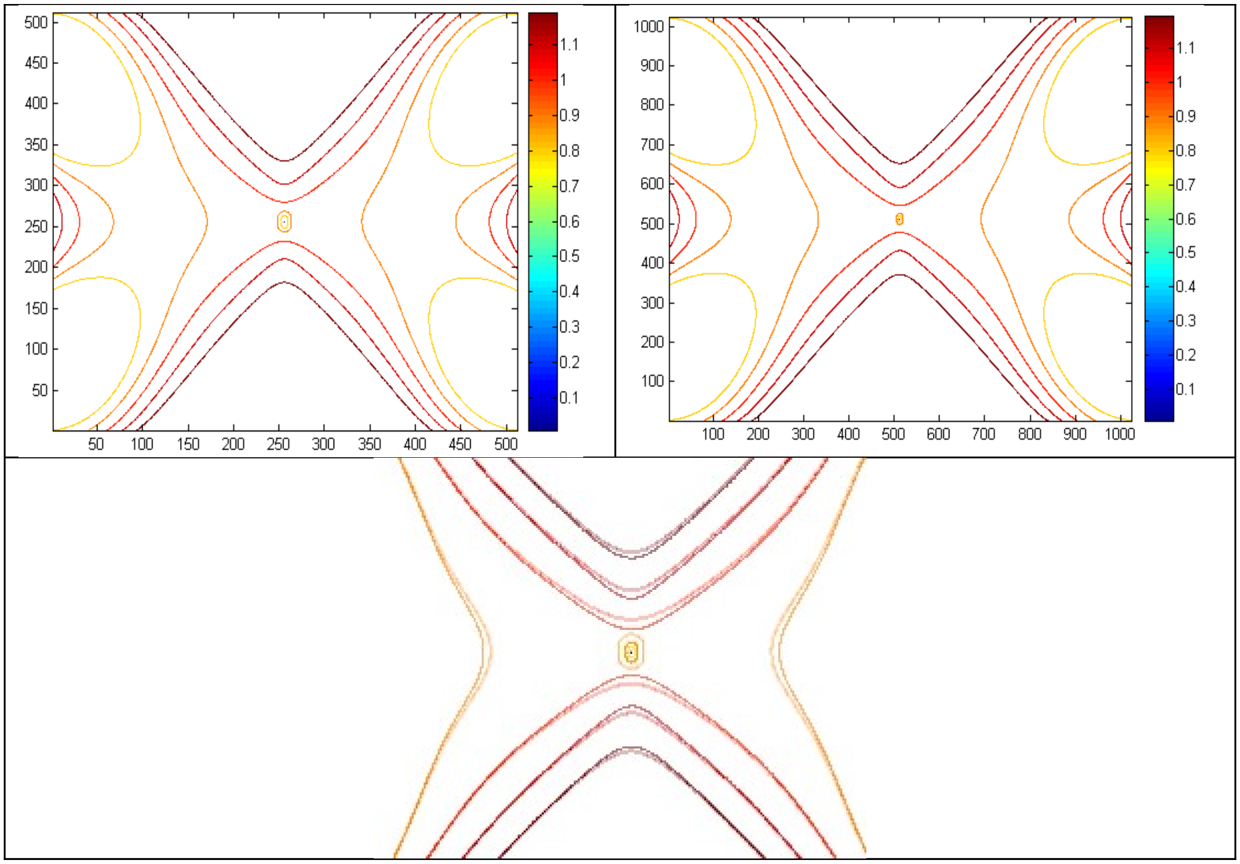

Figure 5. The ratio of numerical shear to analytical values for the Singular Isothermal Ellipse. The left image is for the initially presented shear values and the right image was produced at double resolution. The bottom image shows an overlay of the two images, demonstrating the slight improvement of increased mass resolution.

Similar results are presented for the Singular Isothermal Ellipse(SIE). Figure 5 compares the numerical and analytical results of the FFT calculation for the SIE. The contour lines in Figure 5 show the ratio of numerical shear to analytical shear. The upper right frame was calculated with the same total mass, but with double the resolution of the left image. In both resolution cases, the FFT results in inaccuracies near the boundary, with this effect being more pronounced along the semi-minor axis. For comparison, the lower image in Figure 5 shows the middle portion of the overlay of the two images. Outside of this area, the contour plots are identical. However, this section of the overlay shows that the increased mass resolution does increase the area of near identical numerical and analytical values, since the higher resolution curve is moved towards the grid boundary, shown with the fainter curves. One might notice a second area of inaccuracy in the center of the images. This divergence is expected as the convergence function is singular near the center.

Discussion

Though a common method of calculation in gravitational lensing, convolution of mass with the Fast Fourier Transform has certain problems associated with it. Any sharp edge brings about alternating under- and over-estimating errors. As previously mentioned, this problem is less of an issue in real lensing studies as such a sharp edge is not realistic. However, the grid boundary problem is a systematic calculation issue. This calculation, using the convolution of mass, is based on the convolution function, which assumes a periodic convergence. One solution to this problem is the use of zero padding. Extending and then filling the edges of the convergence with zeros prior to convolution moves the periodic boundary problems to the edge which is no longer part of the mass being studied. The benefits of zero padding are alluded to in the uniform ellipse plots, where most of the grid is in fact zero. However, zero padding the convergence now introduces long wavelength inaccuracies which must be filtered out. Furthermore, the grid size has been increased, often suggested to be double in size, therefore reducing the efficiency of the calculation. Since speed is a motivating factor in using the FFT method, increasing the computational burden while not gaining any resolution should perhaps be an indicator that the FFT method may need to be altered or replaced.

One possible solution is a more direct computation of shear by summing particle contributions to shear and convergence, weighted by proximity. Aubert et al 2007 show promising results with this method. By summing over smoothed particle contributions, increased precision is gained in shear, convergence and deflection angle calculations. One drawback to this method is the calculation time. In contrast to the FFT method, the summation method scales on order of $O(N_p * N_r)$, where N_p is the number of particles and N_r is the number of points being calculated. However, modern computing methods make this less of an issue, with computing clusters becoming more affordable for universities and internet distributing computing bringing free parallel computing to any user with a computer and an internet connection[8].

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Biography
David Coss is currently perusing a Doctorate in Physics from University of Missouri - St Louis. His field of interest is gravitational lensing and its use in studying dark matter.

References
1. Squires, G., Kaiser, N. 1996. ApJ. 473: 65-80.
2. Coss, David. *Simulation of Gravitational Lensing*. 2008. Proceeding from the 2008 NASA/MOSGC Meeting. http://research.davecoss.com/nasa_mo_paper.pdf.
3. Schneider P., Ehlers, J., Falco, E. E. *Gravitational Lenses*. Springer: 1999.
4. Press, W., Teukolsky, S., Vetterling, W., Flannery, B. *Numerical Recipes: The Art of Scientific Computing*. 3rd Edition. 1992. Cambridge University Press.
5. Natarajan, P., Refregier, A. 2000. ApJL. 538: L113-L116.
6. Schramm, T. 1990. A&A. 231: 19-34.
7. Aubert, D., Amara, A., Metcalf, R. 2007. MNRAS. 376: 113-124.
8. Coss, David. "BOINC Distributed Computing." http://research.davecoss.com/boinc/