Various Proofs and Derivations

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1 Complex Numbers

Let $\zeta \in \mathbb{C}$, ξ, $\eta \in \mathbb{R}$ and $\zeta \equiv \xi + \imath\eta$, then:

$$Re(\zeta) = \xi = \frac{1}{2}(\zeta + \overline{\zeta}) \tag{1}$$

$$Im(\zeta) = \eta = \frac{1}{2\imath}(\zeta - \overline{\zeta}) \tag{2}$$

2 Complex Vectors

Let ξ, η, x, $y \in \mathbb{R}$ and

$$\vec{\zeta} \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{3}$$

$$\vec{z} \equiv \begin{pmatrix} x \\ y \end{pmatrix} , \tag{4}$$

which can also be written as vectors in the complex plane,

$$\zeta = \xi + \imath\eta \tag{5}$$

$$z = x + \imath y. \tag{6}$$

2.1 Dot Product

$$\vec{\zeta} \cdot \vec{z} = \frac{1}{2}\left[\zeta\overline{z} + \overline{\zeta}z\right] \tag{7}$$

Proof:

$$\vec{\zeta} \cdot \vec{z} \equiv \xi x + \eta y \tag{8}$$

$$= Re(\zeta)\, Re(z) + Im(\zeta)\, Im(z) \tag{9}$$

$$= \frac{1}{4}\left[(\zeta + \overline{\zeta})(z + \overline{z}) - (\zeta - \overline{\zeta})(z - \overline{z})\right] \quad \text{by Equations (1) \& (2)} \tag{10}$$

$$= \frac{1}{2}\left[\zeta\overline{z} + \overline{\zeta}z\right] \tag{11}$$

The reverse is simpler:

$$\zeta\overline{z} + \overline{\zeta}z = 2\xi x + 2\eta y \tag{12}$$

$$\Rightarrow \frac{1}{2}(\zeta\overline{z} + \overline{\zeta}z) = Re(\zeta)\, Re(z) + Im(\zeta)\, Im(z) \tag{13}$$

$$= \vec{\zeta} \cdot \vec{z} \tag{14}$$

2.2 Cross Product

$$|\vec{\zeta} \times \vec{z}| = \frac{\imath}{2}\left[\zeta\overline{z} - \overline{\zeta}z\right] \tag{15}$$

Proof:

$$|\vec{\zeta} \times \vec{z}| \equiv \xi y - \eta x \tag{16}$$

$$\begin{aligned}
|\vec{\zeta} \times \vec{z}| &= Re(\zeta)\, Im(z) - Re(z)\, Im(\zeta) && (17)\\
&= \frac{-\imath}{4}\left[(\zeta+\overline{\zeta})(z-\overline{z}) - (\zeta-\overline{\zeta})(z+\overline{z})\right] \quad \text{by Equations (1) \& (2)} && (18)\\
&= \frac{i}{2}(\zeta\overline{z} - \overline{\zeta}z) && (19)
\end{aligned}$$

Reverse:

$$\begin{aligned}
\zeta\overline{z} - \overline{\zeta}z &= (\xi + \imath\eta)(x - \imath y) - (\xi - \imath\eta)(x + \imath y) && (20)\\
&= -2\imath\xi y + 2\imath\eta x && (21)
\end{aligned}$$

$$\begin{aligned}
-2i(\xi y - \eta x) &= \frac{2}{i}|\vec{\zeta} \times \vec{z}| \quad \text{by Definition of Cross Product} && (22)\\
\Rightarrow\ |\vec{\zeta} \times \vec{z}| &= \frac{i}{2}(\zeta\overline{z} - \overline{\zeta}z) && (23)\\
& && (24)
\end{aligned}$$

3 Hypergeometric Functions

${}_2F_1(a, b; c; x) -_2 F_1(a-1; b; c; x) = x\frac{b}{c}{}_2F_1(a, b+1; c+1; x)$

Proof:

$$\begin{aligned}
{}_2F_1(a, b; c; x) &\equiv \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} && (25)\\
{}_2F_1(a, b; c; x) -_2 F_1(a-1; b; c; x) &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{\Gamma(b+k)}{\Gamma(c+k)} \{\frac{\Gamma(a+k)}{\Gamma(a)} - \frac{\Gamma(a+k-1)}{\Gamma(a-1)}\} && (26)\\
&= x\frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \frac{\Gamma(b+k)}{\Gamma(c+k)} \{\frac{\Gamma(a+k)}{\Gamma(a)} - \frac{\Gamma(a+k-1)}{\Gamma(a-1)}\} && (27)\\
\frac{\Gamma(a+k)}{\Gamma(a)} - \frac{\Gamma(a+k-1)}{\Gamma(a-1)} &= \frac{\Gamma(a+k)[a+k-1-a+1]}{[a+k-1)\Gamma(a)} && (28)\\
&\quad Let k \to k+1\\
&\Longrightarrow x\frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(b+k+1)\Gamma(a+k)}{\Gamma(c+k+1)\Gamma(a)} && (29)\\
&\Gamma(z+1) \equiv z\Gamma(z) && (30)\\
\Longrightarrow x\frac{b}{c}\frac{\Gamma(c+1)}{\Gamma(a)\Gamma(b+1)} \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{\Gamma(a+k)\Gamma(b+k+1)}{\Gamma(c+k+1)} &\equiv x\frac{b}{c}{}_2F_1(a, b+1; c+1; x) && (31)
\end{aligned}$$

Note: In step 27, the beginning of sum may be changed to k=1 without loss of generality because the zeroth order term equals zero.